Forces, moments, and added masses for Rankine bodies

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SUMMARY

The dynamical theory of the motion of a body through an inviscid and incompressible fluid has yielded three relations: a first, due to Kirchhoff, which expresses the force and moment acting on the body in terms of added masses; a second, initiated by Taylor, which expresses added masses in terms of singularities within the body; and a third, initiated by Lagally, which expresses the forces and moments in terms of these singularities. The present investigation is concerned with generalizations of the Taylor and Lagally theorems to include unsteady flow and arbitrary translational and rotational motion of the body, to present new and simple derivations of these theorems, and to compare the Kirchhoff and Lagally methods for obtaining forces and moments. In contrast with previous generalizations, the Taylor theorem is derived when other boundaries are present; for the added-mass coefficients due to rotation alone, for which no relations were known, it is shown that these relations do not exist in general, although approximate ones are found for elongated bodies. The derivation of the Lagally theorem leads to new terms, compact expressions for the force and moment, and the complete expression of the forces and moments in terms of singularities for elongated bodies.

1. INTRODUCTION

In the decade from 1920 to 1930 there were published by Lagally, Munk, and Taylor a number of hydrodynamic theorems concerning the added masses of bodies moving through an inviscid fluid and the forces and moments acting upon them. These theorems enable the forces, moments, and added masses to be determined when the singularity distributions of sources, sinks, and doublets within the body, which may be considered to generate the potential flow about it, are known. Since, for the important class of elongated bodies, simple approximations to the singularity distributions are given directly in terms of the body shape, these theorems have furnished a powerful means of investigating the forces and moments acting on such bodies, especially near a free surface. Until recently, the scope of the aforementioned theorems has been limited: that of Taylor (1928) to only one kind of added-mass coefficient, and that of Lagally (1922) to steady flow only. Birkhoff (1953, p. 161) and Landweber (1956) have succeeded in generalizing

Taylor's theorem to apply to all the added-mass coefficients except those for pure rotation, and Cummins (1953) in generalizing Lagally's theorem to apply to cases of unsteady flows.

The Taylor theorem and its generalizations have heretofore been concerned with the added-mass coefficients corresponding to the motion of a single body in an otherwise undisturbed and unbounded fluid. The extension of the theorem to the important cases where external singularities and other boundaries are present is highly desirable.

Cummins was able to express the force and moment on a body in terms of the strengths of the singularities, except for one term in the expression for the moment—an integral over the surface of the body with integrand linear in the potential. As will be seen, this single unresolved term is intimately related to the missing relations in the generalization of Taylor's theorem. The discovery of the latter would complete the generalizations of both the Taylor and Lagally theorems.

The present work, then, has several purposes.

(1) The first is to extend the Taylor theorem to include cases with external singularities and boundaries and new results concerning the missing relations between added masses and singularities. The latter will be derived for ellipsoids and for elongated bodies, but it will be proved that such relations do not exist in general. Also the opportunity will be taken to present, new, short, and simple proofs of the theorem for both two and three dimensional flows.

(2) The second is to consolidate and extend Cummins' results and to present a simpler derivation of them. An important secondary motive for this part of the work is to popularize this powerful theorem, which in its present form has been applied only by Cummins himself.

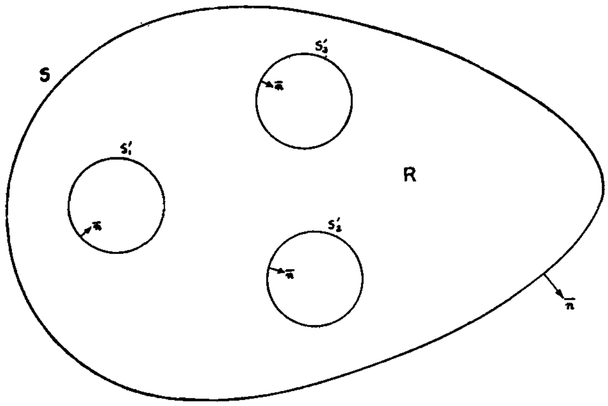
(3) The third is to examine the interconnections, if any, between the Taylor theorem (relating added masses to singularities), the Lagally theorem (relating singularities to forces), and Kirchhoff's equations of motion (relating forces to added masses).

2. Formulation of the problem

We are concerned with the interactions of a fluid with a rigid body moving through it. The fluid is assumed to be incompressible and inviscid, the flow to be irrotational and, in general, unsteady. We shall suppose that the unsteadiness may be due to the time dependence of the linear or angular velocities of the body or to the presence of external boundaries or flow-producing mechanisms which may themselves be moving in an arbitrary manner.

The flow may be considered to be generated by singularities, and the singularities considered will be isolated sources or sinks, doublets, and continuously distributed sources or sinks. Continuous distributions of doublets are excluded from consideration because they can be replaced by corresponding ones of sources and sinks. The symbol m will denote the strength of a source or sink, $\bar{\mu}$ (a vector) will describe both the strength and the orientation of a doublet, and σ will denote the volume or area density of continuously distributed sources or sinks.

Cartesian coordinates (fixed in the body) x_i ($i = 1, 2, 3$) will be used, so that the position vector $\bar{r}$ is (x_1, x_2, x_3) with magnitude r. The components of the velocity vector $\bar{u}$ of the origin of coordinates will be denoted by(u_1, u_2, u_3) and the components of the angular velocity $\bar{\omega}$ will be designated alternatively by $(\omega_1, \omega_2, \omega_3)$ or by (u_4, u_5, u_6). The surface of the body will be denoted by S, and those surrounding the singularities inside S by S' collectively. The distance n normal to either S or S' is always directed out of that portion of the fluid with which one is concerned. The direction of n will be indicated by the unit vector $\bar{n}$ with components n_i ($i = 1, 2, 3$), which are given by $n_i = \partial x_i/\partial n$.



The kinematic boundary condition at a point on S is given by

$$-\frac{\partial \phi}{\partial n} = (\bar{u}+\bar{\omega}\times\bar{r}).\bar{n} = u_\alpha n_\alpha, \tag{1}$$

in which ϕ is the velocity potential satisfying the Laplace equation, and the components of $\bar{r}\times\bar{n}$ are denoted by (n_4, n_5, n_6). Here the summation convention has been adapted and the Greek subscripts range from 1 to 6. For convenience of presentation, Greek subscripts will range from 1 to 6, whereas Latin ones will range over 1, 2, and 3 only, unless otherwise stated or when summation signs are expressly used.

The velocity potential ϕ may be considered to be composed of a part due to the motion of the body alone, when all other boundaries and external flow producing mechanisms are at rest, expressible in the form $u_\alpha \phi_\alpha$ after Kirchhoff, and a part ϕ_0 due to the motions of the latter when the body is at rest, i. e.

$$\phi = u_\alpha \phi_\alpha + \phi_0. \tag{2}$$

Then, on S, from (1) and (2),

$$\frac{\partial \phi_0}{\partial n} = 0, \qquad -\frac{\partial \phi_0}{\partial n} = n_\alpha. \tag{3}$$

The boundary condition (1) may also be expressed in the alternative form

$$v_i n_i = (u_i + \epsilon_{ijk}\,\omega_j\,x_k)n_i,$$

where $v_i = -\partial\phi/\partial x_i$ is the i-component of the velocity at a point of the fluid, and ϵ_{ijk} is zero if any two of the indices i, j, k are alike, and $+1$ or -1 according as the indices are in cyclic or anticyclic order. Hence, the boundary condition becomes

$$V_i n_i = 0, \qquad V_i = -v_i + u_i + \epsilon_{ijk}\,\omega_j\,x_k. \tag{4}$$

Since the coordinate axes are in motion, the Bernoulli equation for the pressure is, from Lamb (1932, p. 20),

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - W, \qquad W = \tfrac{1}{2}(u_j u_j + v_j v_j) - v_i(u_i + \epsilon_{jkl}\,\omega_k\,x_l), \tag{5}$$

where ρ is the density of the fluid.

It will also be convenient to have the expression for the kinetic energy T_B of the displaced fluid, considered as a rigid body. Expansion of the integrand in

$$2T_B = \rho\int(u_i + \epsilon_{ijk}\,\omega_j\,x_k)(u_i + \epsilon_{ilm}\,\omega_l\,x_m)\,d\tau,$$

where $d\tau$ denotes an element of volume of the body, readily yields the quadratic form

$$2T_B = B_{\alpha\beta}\,u_\alpha\,u_\beta, \tag{6}$$

$$\left.\begin{aligned} B_{\alpha\beta} = B_{\beta\alpha}, \qquad B_{ij} = B\delta_{ij}, \qquad B_{i,3+j} = B\epsilon_{ijk}\bar{x}_k, \\ B_{3+j,3+k} = \rho\int(x_m\,x_m\,\delta_{jk} - x_j\,x_k), \end{aligned}\right\} \tag{7}$$

where B is the mass of the displaced fluid, δ_{ij} is the Kronecker delta, and $\bar{x}_k$ is the k-component of the centroid of volume in the body.

3. Mathematical preliminaries

In this section will be collected various mathematical theorems and results which will be required in the subsequent sections.

3.1. *Potential functions*

Let ϕ be a potential function which satisfies Laplace's equation at points where no singularities are present, and which, in regions where there is a source distribution of strength σ, satisfies Poisson's equation

$$\frac{\partial^2\phi}{\partial x_i\,\partial x_i} = -4\pi\sigma. \tag{8}$$

In the neighbourhood of a point source of strength m at the point with coordinates x_{is}, the potential may be expressed in the form

$$\phi = \phi' + \frac{m}{r_s}, \qquad r_s^2 = (x_i - x_{is})(x_i - x_{is}), \tag{9}$$

or, since ϕ' is analytic in the neighbourhood of x_{is},

$$\phi = \frac{m}{r_s} + (\phi')_s + (x_i - x_{is})\left(\frac{\partial\phi'}{\partial x_i}\right)_s + \ldots,$$

and similarly the velocity field is expressible in the form

$$v_i = -\frac{mn_i}{r_s^2} + (v_i')_s + (x_j - x_{js})\left(\frac{\partial v_i'}{\partial x_j}\right)_s + \dots, \tag{11}$$

$$n_i = \frac{x_{is} - x_i}{r_s}, \qquad v' = -\frac{\partial \phi'}{\partial x_i}. \tag{12}$$

In the neighbourhood of a point doublet of vector strength $\bar{\mu}$ at the point x_{id}, the potential may be expressed in the form

$$\phi = \phi' - \frac{\mu_i n_i}{r_d^2}, \qquad n_i = \frac{x_{id} - x_i}{r_d}, \qquad r_d^2 = (x_i - x_{id})(x_i - x_{id}), \tag{13}$$

or, since ϕ' is analytic in the neighbourhood of x_{id},

$$\phi = -\frac{\mu_i n_i}{r_d^2} + (\phi')_d + (x_i - x_{id})\left(\frac{\partial \phi'}{\partial x_i}\right)_d + \dots, \tag{14}$$

$$v_i = \frac{1}{r_d^3}(3\mu_j n_i n_j - \mu_i) + (v_i')_d + (x_j - x_{jd})\left(\frac{\partial v_i'}{\partial x_j}\right)_d + \dots \tag{15}$$

3.2. *Gauss's and Green's theorems*

Let $\phi(x_1, x_2, x_3)$ be a function analytic in a region R and on its boundaries, where the region R is bounded externally by a closed surface S and internally by a set of spheres whose surfaces are collectively designated by S'. The sense of the normals to the boundaries outward from the region R is taken as positive.

We can now state Gauss's theorem in the form

$$\int \phi n_i \, dS = \int \frac{\partial \phi}{\partial x_i} d\tau - \int \phi n_i \, dS', \tag{16}$$

where the integral on the left extends over the surface of the outer boundary, the last integral over the surface of the spheres, and the first integral on the right is a volume integral over the region R. Also, if $\psi(x_1, x_2, x_3)$ is another function analytic in R and on its boundaries, we can state the second of Green's theorems in the form

$$\int\left(\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right) dS = \int\left(\phi\frac{\partial^2 \psi}{\partial x_i \partial x_i} - \psi\frac{\partial^2 \phi}{\partial x_i \partial x_i}\right) d\tau - \int\left(\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right) dS',$$

or, if ϕ satisfies Poisson's equation and ψ Laplace's equation,

$$\int\left(\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right) dS = 4\pi \int \psi\sigma \, d\tau - \int\left(\phi\frac{\partial \psi}{\partial n} - \psi\frac{\partial \phi}{\partial n}\right) dS'. \tag{17}$$

3.3 *Boundary conditions and the Bernoulli equation*

Various partial derivatives of the quantities V_i and W defined in (4) and (5) will be required. These are given in this section.

We readily obtain

$$\frac{\partial V_i}{\partial x_k} = -\frac{\partial v_i}{\partial x_k} + \epsilon_{ijk}\,\omega_j, \tag{18}$$

If ϕ satisfies Poisson's equation, we have, when $k = i$,

$$\frac{\partial V_i}{\partial x_i} = -4\pi\sigma. \tag{19}$$

Next, if ϕ satisfies Poisson's equation, noting that $\partial v_i/\partial x_j = \partial v_j/\partial x_i$, and applying (19), we obtain

$$\frac{\partial W}{\partial x_i} = \frac{\partial}{\partial x_j}(-v_i V_j + \epsilon_{ijk}\,\phi\omega_k) - 4\pi\sigma v_i. \tag{20}$$

Then, from (27) we obtain

$$\epsilon_{ijk}\,x_j\frac{\partial W}{\partial x_k} = \epsilon_{ijk}\left[-\frac{\partial}{\partial x_l}(x_j v_k V_l) + V_j v_k - 4\pi\sigma x_j v_k - \epsilon_{klm}\,x_j v_l\,\omega_m\right].$$

But, substituting for V_j, and simplifying, we have

$$\epsilon_{ijk}(V_j v_k - \epsilon_{klm}\,x_j v_l\,\omega_m) = \epsilon_{ijk}(u_i v_k + \epsilon_{klm}\,\omega_j\frac{\partial\phi}{\partial x_l}x_m).$$

Hence

$$\epsilon_{ijk}\,x_j\frac{\partial W}{\partial x_k} = \epsilon_{ijk}\left[-\frac{\partial}{\partial x_l}(x_j v_k V_l) - u_j\frac{\partial\phi}{\partial x_k} - 4\pi\sigma x_j v_k + \epsilon_{klm}\,\omega_j\frac{\partial\phi}{\partial x_l}x_m\right]. \tag{21}$$

4. Added masses

If we consider the kinetic energy T of the fluid due to the motion of the body when all the other boundaries and flow-producing mechanisms are at rest, we have the well-known formula

$$2T = -\rho\int\phi\frac{\partial\phi}{\partial n}\,dS. \tag{22}$$

From (1) and (2) it follows that

$$2T = A_{\alpha\beta}\,u_\alpha\,u_\beta, \tag{23}$$

in which the added masses $A_{\alpha\beta}$ are given by

$$A_{\alpha\beta} = \rho\int\phi_\alpha\,n_\beta\,dS, \qquad A_{\alpha\beta} = A_{\beta\alpha}. \tag{24}$$

Can the added masses be expressed simply in terms of the singularities inside the body under consideration as some of them ($A_{\alpha i}$) were by Taylor, Birkhoff, and Landweber for a body moving in unbounded fluid? This is the chief concern of this section. In the following, two-dimensional and three-dimensional flows will be discussed separately. Two dimensional flows are discussed not only because the use of function-theoretic methods enables one to solve the problem for two dimensions from an entirely new approach and in a singularly simple manner, but also because *simple* relationships between $A_{\alpha\beta}$ and the singularities can be shown to be non-existent for pure rotations.

4.1. *Two-dimensional flows*

The complex variable z is defined, as usual, to be $x_1 + ix_2$. With every velocity potential ϕ_α ($\alpha = 1, 2, 6$ throughout the discussion for two-dimensional flows) one may associate a stream function ψ_α such that the complex potential is $w_\alpha = \phi_\alpha + i\psi_\alpha$. Now, if s denotes the curvilinear distance along the body, we have $n_1\,ds = dx_2$, $n_2\,ds = -dx_1$, and since

$$\frac{\partial\psi_\alpha}{\partial s} = \frac{\partial\phi_\alpha}{\partial n} = -n_\alpha, \tag{25}$$

$$A_{\alpha1} + iA_{\alpha2} = \rho \oint \phi_\alpha n_1\,ds + i\rho \oint \phi_\alpha n_2\,ds = i\rho \oint \phi_\alpha\,dz$$

$$= -i\rho\left[\oint w_\alpha\,dz - i\oint \psi_\alpha\,dz\right] = -i\rho\oint w_\alpha\,dz - \rho\oint zn_\alpha\,ds.$$

Thus,

$$A_{\alpha1} + B_{\alpha1} + i(A_{\alpha2} + B_{\alpha2}) = -i\rho\oint w_\alpha\,dz, \tag{26}$$

in which

$$B_{\alpha1} = \rho\oint x_1 n_\alpha\,ds, \qquad B_{\alpha2} = \rho\oint x_2 n_\alpha\,ds. \tag{27}$$

By Gauss's theorem, we have

$$B_{ij} = B\delta_{ij}, \quad B_{61} = -B\bar{x}_2, \quad B_{62} = B\bar{x}_1, \tag{28}$$

where B is the mass of the displaced fluid per unit length of the body, and x_1, x_2 are the coordinates of the centroid of its area of section.

Equation (26) readily yields the generalized Taylor theorem. If there is a doublet of strength μ_α inside S, where in general μ_α is a complex number, there is a term $\mu_\alpha/(z - z_d)$ in w, and its contribution to the integral in (26) is $2\pi\mu_\alpha i$, by Cauchy's theorem; and if there is a source of strength m_α at $z_{\alpha s}$ inside S, there is a term $-m_\alpha \log(z - z_s)$ in w_α, and its contribution to the same integral is $2\pi m_\alpha z_s i$. Thus, extending the result to distributed sources and sinks of strength σ_α per unit area of section, we have

$$A_{\alpha1} + B_{\alpha1} + i(A_{\alpha2} + B_{\alpha2}) = 2\pi\rho\left[\int \sigma_\alpha z\,dA + \Sigma\,(m_\alpha z_s + \mu_\alpha)\right], \tag{29}$$

in which dA is an element of the area of section over which the integral extends. Equation (28) gives precisely the generalized Taylor theorem for two-dimensional flows.

What, then, about the added mass for pure rotation? The added mass in question is

$$A_{66} = \rho\oint \phi_6 n_6\,ds = \rho\oint \phi_6(x_1 n_2 - x_2 n_1)\,ds = -\rho\oint \phi_6(x_1\,dx_1 + x_2\,dx_2)$$

$$= -\rho\mathscr{R}\left\{\oint w_6 z^* dz\right\} + \rho\oint \psi_6(x_2\,dx_1 - x_1\,dx_2),$$

where z^* is the complex conjugate of z. But, putting

$$F_1(s) = \int_0^s x_1 n_1\,ds, \qquad F_2(s) = \int_0^s x_2 n_2\,ds,$$

integrating by parts, and applying (25), we have

$$\oint \psi_6(x_2\,dx_1 - x_1\,dx_2) = -\oint \psi_6(x_1 n_1 + x_2 n_2)\,ds$$

$$= -\oint (x_1 n_2 - x_2 n_1)(F_1 + F_2)\,ds - \psi_6(0) \oint (x_1 n_1 + x_2 n_2)\,ds$$

$$= \tfrac{1}{2}[(x_1^2 + x_2^2)_0 - 2\psi_6(0)] \oint (x_1 n_1 + x_2 n_2)\,ds$$

$$-\tfrac{1}{2} \oint (x_1^2 + x_2^2)(x_1 n_1 + x_2 n_2)\,ds,$$

where the index 0 denotes initial values along the path of integration. Applying Gauss's theorem, we obtain

$$\oint \psi_6(x_2\,dx_1 - x_1\,dx_2) = A[(x_1^2 + x_2^2)_0 - 2\psi_6(0)] - 2\int (x_1^2 + x_2^2)\,dA.$$

We may take $\psi_6(0) = 0$. Also, let r_g be the radius of gyration of the area, and let $r_0^2 = (x_1^2 + x_2^2)_0$. Then we have, finally,

$$A_{66} + B_{66} = \rho A(r_0^2 - r_g^2) - \rho \mathscr{R}\left\{\oint w_6\, z^* dz\right\}. \tag{30}$$

It is now seen, by expressing w_6 in terms of its singularities, that (30) gives a linear relation between A_{66} and their strength. If, for instance, among the singularities of w_6, there are doublets of strength μ_i at $z = z_i$, each would give rise to a term

$$-\rho\mu_i \mathscr{R}\left\{\oint \frac{z^*}{z - z_i}\,dz\right\}$$

which, however, cannot be expressed in the form $\mu_i f(z_1, z_2, \ldots)$ independent of the shape of the profile; for otherwise, as a consequence of Morera's theorem of complex variables, the integrand would be an analytic function, which it clearly is not. In this sense a simple relationship like those of (29) does not exist for A_{66}. The writers have laboured much and in vain in their endeavour to establish the missing simple relationships for pure rotations, and it was not until equation (30) was reached that they recognized the futility of their efforts.

4.2. *Three-dimensional flows*

For the fluid inside S and outside the small surface S' around the interior singularities, we can apply (17) to obtain

$$A_{j\alpha} = A_{\alpha j} = \rho \int \phi_\alpha n_j\,dS = \rho \int \phi_\alpha \frac{\partial x_j}{\partial n}\,dS$$

$$= \rho \int x_j \frac{\partial \phi_\alpha}{\partial n}\,dS + 4\pi\rho \int x_j \sigma_\alpha\,d\tau + \rho \int \left(x_j \frac{\partial \phi_\alpha}{\partial n} - \phi_\alpha n_j\right) dS', \tag{31}$$

where σ_α is the distributed source strength corresponding to ϕ_α. Here, as shown by Landweber (1956),

$$\rho \int x_j \frac{\partial \phi_\alpha}{\partial n}\,dS = -\rho \int x_j n_\alpha\,dS = -B_{\alpha j}. \tag{32}$$

Moreover, for the source,

$$\int x_j \frac{\partial \phi_\alpha}{\partial n} dS' = 4\pi \sum_s m_\alpha x_{js}. \tag{33}$$

For the doublet μ_α, with components $\mu_{\alpha i}$, the potential is given by (13), from which, writing

$$x_j \frac{\partial \phi_\alpha}{\partial n} = x_j \frac{\partial \phi'_\alpha}{\partial n} - \frac{2x_{jd}\, \mu_{\alpha i} n_i}{r_d^3} + \frac{2\mu_{\alpha i} n_i n_j}{r_d^2},$$

and noting that $\int x_j(\partial \phi'/\partial n)\, dS'$ vanishes in the limit as r_d approaches zero, and $\int n_i r_d^{-3} dS'$ and $\int n_i n_j r_d^{2}\, dS'$ vanish because of anti-symmetry except when $i = j$ in the last integral, we have

$$\int x_j \frac{\partial \phi_\alpha}{\partial n} dS' = 2\mu_{\alpha j} \int \frac{n_j n_j}{r_d^2} dS' = \frac{4\pi}{3} \mu_{\alpha j}, \tag{34}$$

where j is not summed in the second member of (34). Next, considering $-\int \phi_\alpha n_i\, dS'$, the contribution to it by a source is zero, since its potential varies inversely as the radial distance r_s from it, whereas dS varies as r_s^2. The contribution of a doublet is

$$-\int \left(\phi' - \frac{\mu_{\alpha_i} n_i}{r_d^2}\right) n_j\, dS' = \mu_{\alpha j} \int \frac{n_j n_j}{r_d^2}\, dS' = \frac{4\pi}{3} \mu_{\alpha j}. \tag{35}$$

Substituting (32) to (35) into (31), we obtain the generalized Taylor theorem

$$A_{\alpha,} + B_{\alpha j} = 4\pi\rho[\textstyle\int \sigma_\alpha x_j\, d\tau + \sum (m_\alpha x_{js} + \mu_{\alpha j})]. \tag{36}$$

4.3. *Evaluation of 'missing' added-mass relations for an elongated body*

In this section simple expression for the added-mass coefficients for pure rotation in terms of the singularities within the body, will be derived for ellipsoids, and approximate ones for elongated bodies.

From (17) we have

$$\int \phi(x_j n_k + x_k n_j)\, dS = \int \phi \frac{\partial}{\partial n}(x_j x_k)\, dS$$

$$= \int x_j x_k \frac{\partial \phi}{\partial n} dS + 4\pi \int \sigma x_j x_k\, d\tau - \int \left[\phi(x_j n_k + x_k n_j) - x_j x_k \frac{\partial \phi}{\partial n}\right] dS'$$

$$= \int x_j x_k \frac{\partial \phi}{\partial n}\, dS + 4\pi \left[\int (\sigma x_j x_k\, d\tau + \sum (m x_j x_k + \mu_j x_k + \mu_k x_j)\right], \tag{37}$$

where the integrals over the spherical surfaces about the isolated singularities have been evaluated by a now familiar process and the subscripts s and d on the coordinates have been omitted in the last terms. Also for the first integral on the right in (37), we readily obtain from (3) and (16) the matrix of values

$$\left.\begin{aligned} \rho \int x_j x_k \frac{\partial \phi_\alpha}{\partial n} dS &= B'_{3+i,\alpha} \quad (\alpha = 4, 5, 6), \\ B'_{3+i,3+i} &= -\rho \epsilon_{ijk} \int (x_j^2 - x_k^2)\, d\tau, \\ B'_{3+j,3+k} &= \rho \epsilon_{ijk} \int x_j x_k\, d\tau, \end{aligned}\right\} \tag{38}$$

where i, j, k are different and not summed. Thus we may write

$$\left.\begin{aligned} \rho \int \phi_\alpha(x_j n_k + x_k n_j)\, dS &= B'_{3+i,\alpha} + 4\pi\rho\Sigma_{jk\alpha}, \\ \Sigma_{jk\alpha} &= \int \sigma_\alpha x_j x_k \, d\tau + \Sigma\, (m_\alpha x_j x_k + \mu_{\alpha j} x_k + \mu_{\alpha k} x_j). \end{aligned}\right\} \tag{39}$$

Also we have

$$A_{\alpha,3+i} = \rho \int \phi_\alpha n_{3+i}\, dS = \rho\epsilon_{ijk} \int \phi_\alpha x_j n_k \, dS. \tag{40}$$

Consider the case of an ellipsoid rotating in an infinite fluid in which no other boundaries or singularities are present. Then we have, from Lamb (1932, p. 154), when the coordinate axes are taken along the principal axes,

$$\phi_4 = C_4 x_2 x_3, \qquad \phi_5 = C_5 x_3 x_1, \qquad \phi_6 = C_6 x_1 x_2, \tag{41}$$

where C_4, C_5, C_6 are constants, and hence, from (39), (40), and (16), for the case $\alpha = 6$, $i = 3$, we obtain

$$\begin{aligned} C_6 B_{66} &= B'_{66} + 4\pi\rho\Sigma_{126} \\ A_{66} &= -C_6 B'_{66}, \end{aligned} \tag{42}$$

or eliminating C_6,

$$A_{66} \cdot \frac{B_{66}}{B'_{66}} + B'_{66} = -4\pi\rho\Sigma_{126}. \tag{43}$$

Similar expressions may be written by symmetry for A_{55} and A_{44}. The coefficients A_{56}, A_{64}, A_{45} are zero for the chosen orientation of the coordinate axes. Thus we have found simple relations between the rotational added-mass coefficients and the singularities for ellipsoids, albeit not as simple as those in (36).

When the x_1-axis of the ellipsoid is much greater than the others, and the x_2- and x_3-axes are nearly equal C_6 is very nearly 1, $A_{66} \doteqdot -B'_{66}$ from (42), and the left member of (43) may be written as

$$A_{66} - A_{66}\left(\frac{B_{66}}{B'_{66}} + 1\right) - B'_{66} \doteqdot A_{66} + B'_{66}\left(\frac{B_{66}}{B'_{66}} + 1\right) - B'_{66} = A_{66} + B_{66},$$

which is of the form that an extension of (36) would suggest.

Next let us consider a body elongated in the direction of the x_1-axis with nearly elliptical sections in the planes $x_1 =$ constant. From (39) we obtain directly and exactly

$$A_{66} - B'_{66} + 2\rho \int \phi_6 x_2 n_1 \, dS = 4\pi\rho\Sigma_{126}. \tag{44}$$

For an elongated body the third term on the left is small compared with the other terms, so that only a small error would be introduced by assuming an approximate value for ϕ_6 in that term. Since the section is nearly elliptical, let us assume $\phi_6 \doteqdot C x_1 x_2$. Then

$$2 \int \phi_6 x_2 n_1 \, dS \doteqdot 2C \int x_2^2 \, d\tau \doteqdot -2\frac{A_{66}}{B'_{66}} \int x_2^2 \, d\tau,$$

whence, substituting into (44) and simplifying, we obtain

$$A_{66} \frac{B_{66}}{B'_{66}} + B'_{66} \doteqdot -4\pi\rho\Sigma_{126}. \tag{45}$$

The corresponding formulae for A_{44} and A_{55} can be written down from symmetry. As in the case of the elongated ellipsoid, when the section is nearly circular we have also

$$A_{66}+B_{66} \doteqdot 4\pi\rho\, \Sigma_{126} \tag{46}$$

and similarly for A_{55}. For A_{44} the term corresponding to the integral in (44) is of the same order of magnitude as the others, so that the errors in the foregoing approximations would be greater than for A_{55} or A_{66}.

Similarly we have, from (39),

$$A_{56}-B'_{56}+2\rho \int \phi_6\, x_1\, n_3\, dS = 4\pi\rho\Sigma_{316}\,, \tag{47}$$

in which the third term may be expressed approximately in the form

$$2\rho \int \phi_6\, x_1\, n_3\, dS \doteqdot 2\rho C \int x_1^2\, x_2\, n_3\, dS = 0.$$

Hence (47) becomes

$$A_{56}+B_{56} \doteqdot 4\pi\rho\, \Sigma_{316}. \tag{48}$$

An alternative relation for A_{56} may be derived by interchanging the roles of the indices, and similar expressions may be obtained for A_{64} and A_{45}. These as well as the relations for A_{44}, A_{55}, and A_{66} may be summarized in the form

$$\left.\begin{aligned} A_{\alpha\alpha}\frac{B_{\alpha\alpha}}{B'_{\alpha\alpha}} + B'_{\alpha\alpha} \doteqdot -(A_{\alpha\alpha}+B_{\alpha\alpha}) &= -4\pi\rho\, \Sigma_{jk\alpha} \quad (\alpha = 3+i\neq 4), \\ A_{\beta\gamma}+B_{\beta\gamma} \doteqdot 4\pi\rho\epsilon_{ijk}\, \Sigma_{ik\gamma} \quad (\beta = 3+j,& \quad \gamma = 3+k), \end{aligned}\right\} \tag{49}$$

in which i, k are not summed and i, j, k are different.

4.4 *Linear and Angular Momentum*

It will be of interest to express the integrals $\rho \int \phi n_i\, dS$ and $\rho\epsilon_{ijk} \int \phi x_j n_k\, dS$ in terms of the added-mass coefficients and the singularities within the body. By applying Gauss's transformation to the region exterior to the body it is seen that the sums of such integrals over all the boundaries give the linear and angular momenta of the fluid.

The velocity potential, in the form given by (4), may be further resolved by writing

$$\phi_0 = \phi'_0+\phi''_0, \tag{50}$$

where ϕ'_0 is the potential of the part of ϕ_0 due to external singularities, and ϕ''_0 that due to internal singularities. We have then, from (24),

$$\rho \int \phi n_i\, dS = u_\alpha\, A_{\alpha i}+\rho \int (\phi'_0+\phi''_0)n_i\, dS. \tag{51}$$

But, by Green's reciprocal theorem and (17),

$$\int \phi'_0\, n_i\, dS = \int x_i \frac{\partial\phi'_0}{\partial n} dS, \tag{52}$$

$$\int \phi''_0\, n_i\, dS = \int x_i \frac{\partial\phi''_0}{\partial n} dS+4\pi \int \sigma_0\, x_i\, d\tau+ \int\left(x_i \frac{\partial\phi''_0}{\partial n} - \phi''_0\, n_i\right) dS', \tag{53}$$

where σ_0 is the strength of the distributed singularities within the body corresponding to ϕ_0. Thus, substituting (52) and (53) into (51), applying the boundary condition (3), and noting that the last integral over the spheres about the singularities is of the same form as that evaluated in (31), we obtain

$$\rho \int \phi n_i \, dS = u_\alpha A_{\alpha i} + 4\pi\rho\left[\int \sigma_0 x_i \, d\tau + \Sigma \, (m_0 x_i + \mu_0)\right], \tag{54}$$

where m_0 and μ_{0i} are the strengths of the sources and doublet components within the body corresponding to ϕ_0. Hence, applying (36), we obtain

$$\rho \int \phi n_i \, dS + u_\alpha B_{\alpha i} = 4\pi\rho\left[\int \sigma x_i \, d\tau + \Sigma \, (m x_i + \mu_i)\right], \tag{55}$$

where σ, m, and μ_i denote the totality of all the distributions and singularities within the body. This shows that the value of the momentum integral depends simply upon the internal singularities.

Next let us consider the angular momentum integral. Recalling the definition $n_{3+i} = \epsilon_{ijk} x_j n_k$, we have, putting $\beta = 3+i$, and applying (24) and (3),

$$\begin{aligned} \rho\epsilon_{ijk} \int \phi x_j n_k \, dS &= \rho \int (u_\alpha \phi_\alpha + \phi_0) n_\beta \, dS \\ &= u_\alpha A_{\alpha\beta} - \rho \int (\phi_0' + \phi_0'') \frac{\partial \phi_\beta}{\partial n} \, dS. \end{aligned} \tag{56}$$

But, from (17), we have

$$\int \phi_0' \frac{\partial \phi_\beta}{\partial n} \, dS = \int \phi_\beta \frac{\partial \phi_0'}{\partial n} \, dS - 4\pi \int \sigma_\beta \phi_0' \, d\tau + \int \left(\phi_\beta \frac{\partial \phi_0'}{\partial n} - \phi_0' \frac{\partial \phi_\beta}{\partial n}\right) dS'.$$

Also, putting $\phi_\beta = \phi_\beta' + \phi_\beta''$, where ϕ_β' is the part of ϕ_β due to external singularities, and ϕ_β'' that due to the internal ones, we have from (17) and Green's reciprocal theorem

$$\int \phi_0'' \frac{\partial \phi_\beta'}{\partial n} \, dS = \int \phi_\beta' \frac{\partial \phi_0''}{\partial n} \, dS + 4\pi \int \sigma_0 \phi_\beta' \, d\tau + \int \left(\phi_\beta' \frac{\partial \phi_0''}{\partial n} - \phi_0'' \frac{\partial \phi_\beta'}{\partial n}\right) dS',$$

$$\int \phi_0' \frac{\partial \phi_\beta''}{\partial n} \, dS = \int \phi_\beta'' \frac{\partial \phi_0''}{\partial n} \, dS,$$

whence, applying the boundary condition (3), evaluating the integrals over S' by the usual procedure, and substituting into (56), we obtain

$$\begin{aligned} \rho\epsilon_{ijk} \int \phi x_j n_k \, dS = u_\alpha A_{\alpha\beta} + 4\pi\rho\Big[\int (\sigma_\beta \phi_0' - \sigma_0 \phi_\beta') \, d\tau + \\ + \Sigma \, (m_\beta \phi_0' - m_0 \phi_\beta' - \mu_{\beta j} v_{0j}' + \mu_{0j} v_{\beta j}')\Big] \end{aligned} \tag{57}$$

or, substituting for $A_{j\beta}$ from (36), and applying (7),

$$\begin{aligned} \rho\epsilon_{ijk} \int \phi x_j n_k \, dS = B\epsilon_{ijk} u_j \bar{x}_k + 4\pi\rho\Big\{\int [\sigma_\beta(\phi_0' + u_j x_j) - \sigma_0 \phi_\beta'] \, d\tau + \\ + \Sigma \, [m_\beta(\phi_0' + u_j x_j) - m_0 \phi_\beta - \mu_{\beta j}(v_{0j} - u_j) + \mu_{0j} v_{\beta j}']\Big\} + \omega_j A_{3+i, 3+j}. \end{aligned} \tag{57 a}$$

It will usually be convenient to choose the origin of coordinates so that the term $B\epsilon_{ijk}\,u_j\,\bar{x}_k$ vanishes, as is done in the equations of rigid body dynamics, where this term also occurs.

For the elongated bodies considered in the previous section we can substitute the approximate values given in (49)for $A_{3+i,\,3+j}$. Thus, applying the simpler of the approximate relations for A_{66}, we obtain

$$\rho\epsilon_{3jk}\int \phi x_j\,n_k\,dS = B\epsilon_{3jk}\,u_j\,\bar{x}_k-\omega_j\,B_{3+j,6}+4\pi\rho\Big[\int(\sigma_6\,\Omega_6-\sigma_0\,\phi'_\beta)\,d\tau+$$

$$+\Sigma\,\Big(m_6\,\Omega_6-m_0\,\phi'_\beta+\mu_{6j}\frac{\partial\Omega_6}{\partial x_j}+\mu_{0j}\,v'_{\beta j}\Big)\Big], \qquad (57\,\mathrm{b})$$

$$\Omega_6 = \phi'_0+u_j\,x_j-\omega_1\,x_2\,x_3+\omega_2\,x_3\,x_1-\omega_3\,x_1\,x_2,$$

and a similar expression for $i = 2$, with

$$\Omega_5 = \phi'_0+u_j\,x_j+\omega_1\,x_2\,x_3-\omega_2\,x_3\,x_1-\omega_3\,x_1\,x_2.$$

For the case $i = 1$, the simpler approximation for A_{44} cannot be used so that the resulting expression for the moment of momentum integral would be more complex in form.

5. The generalized Lagally theorem

5.1. *Force on the body*

From (5), the force on the body is given by

$$F_i = -\int pn_i\,dS = -\rho\int\Big(\frac{\partial\phi}{\partial t}-W\Big)n_i\,dS$$

$$= -\rho\frac{d}{dt'}\int\phi n_i\,dS+\rho\int Wn_i\,dS, \qquad (58)$$

where the prime in t' denotes that the variation with time is relative to a moving coordinate system. But, from (16) and (20),

$$\int Wn_i\,dS = \int\frac{\partial W}{\partial x_i}\,d\tau-\int Wn_i\,dS'$$

$$= \int\frac{\partial}{\partial x_j}(-v_i\,V_j+\epsilon_{ijk}\,\phi\omega_k)\,d\tau-4\pi\int\sigma v_i\,d\tau-\int Wn_i\,dS'.$$

Also, applying (4) and (16), we have

$$\int\frac{\partial}{\partial x_j}(-v_i\,V_j+\epsilon_{ijk}\,\phi\omega_k)\,d\tau = \epsilon_{ijk}\,\omega_k\int\phi n_j\,dS+\int(-v_i\,V_j+\epsilon_{ijk}\,\phi\omega_k)n_j\,dS'.$$

Hence, substituting into (58), we obtain

$$F_i = -\rho\frac{d}{dt'}\int\phi n_i\,dS+\rho\epsilon_{ijk}\,\omega_k\int\phi n_j\,dS-4\pi\rho\int\sigma v_i\,d\tau+$$

$$+\rho\int[(-v_i\,V_j+\epsilon_{ijk}\,\phi\omega_k)n_j-Wn_i]\,dS'. \quad (59)$$

The sum of the first two terms of (59) is seen to be the absolute time derivative of the momentum vector integral $-\rho\int\phi n_i\,dS$ for which values were given in (54) and (55). Thus, evaluating the integrals over S' by the usual procedure, we obtain the expression for the force

$$F_i = -\frac{d}{dt}(u_\alpha A_{\alpha i}) - 4\pi\rho\frac{d}{dt}\left[\int \sigma_0 x_i\, d\tau + \Sigma\,(m_0 x_i + \mu_{0i})\right] - $$
$$ -4\pi\rho\left[\int \sigma v_i\, d\tau + \sum\left(m v_i' - \tfrac{4}{3}\pi\sigma\mu_i + \mu_j\frac{\partial v_i'}{\partial x_j}\right)\right], \quad (60)$$

or, if (55) is used in (59), and (7) is applied to evaluate

$$u_\alpha B_{\alpha i} = u_j B_{ji} + \omega_j B_{3+j,i} = B(u_i + \epsilon_{ijk}\,\omega_j\,\bar{x}_k) = B\bar{u}_i\,,$$

where $\bar{u}_i$ is the i-component of the velocity of the centroid, then

$$F_i = B\frac{d\bar{u}_i}{dt} - 4\pi\rho\frac{d}{dt}\left[\int \sigma x_i\, d\tau + \Sigma(m x_i + \mu_i)\right] - $$
$$ -4\pi\rho\left[\int \sigma v_i\, d\tau + \sum\left(m v_i' - \tfrac{4}{3}\pi\sigma\mu_i + \mu_j\frac{\partial v_i'}{\partial x_j}\right)\right]. \quad (61)$$

If it is desired to use relative rather than absolute time derivatives, we may employ alternative expressions of which a typical one is

$$\frac{d}{dt}(\sigma x_i) = \frac{d}{dt'}(\sigma x_i) + \sigma\epsilon_{ijk}\,\omega_j\,x_k.$$

Furthermore, it should be noted that the strengths of the singularities occurring in (60) and (61) are those due to the superimposed effects of all the velocity components of both the body and external boundaries and flow-producing mechanisms, in contrast with the singularity strengths corresponding to unit magnitude of a single velocity component of the body which occur in the generalized Taylor formulas for the added masses.

It is important to observe that, in computing v_i' and $\partial v_i'/\partial x_j$ in (60) and (61), the contributions from all the internal singularities may be omitted. The reason for this is that the mutual contributions of the velocity fields of a pair of internal singularities to the expression for the force on the body are equal and opposite and hence annul each other. This introduces a significant simplification in the calculation of the force from these equations.

Equation (61) is essentially equivalent to the corresponding result by Cummins (1953). It differs from it in the following respects.

(1) A new term, $16\pi^2\rho\sigma\mu_i/3$, has appeared. This did not occur in the treatments of Lagally or Cummins because they did not consider the case of simultaneous occurrence of distributed and isolated singularities.

(2) The inertia term in Cummins' equation (8) and the terms of his equation (48) for the vector F_3 have been replaced by the first term of (61), which is seen to represent the inertia of the displaced fluid.

(3) The expression for the force has been expressed in a much more compact form.

5.2. *Moment on the Body*

The moment on the body is given, from (5), by

$$M_i = -\epsilon_{ijk}\int p x_j n_k \, dS = -\rho\epsilon_{ijk}\int\left(\frac{\partial\phi}{\partial t} - W\right)x_j n_k \, dS$$
$$= -\rho\epsilon_{ijk}\left[\frac{d}{dt'}\int \phi x_j n_k \, dS - \int W x_j n_k \, dS\right]. \tag{62}$$

From (16), (21), and (4), we obtain

$$\epsilon_{ijk}\int W x_j n_k \, dS = \epsilon_{ijk}\left[\int x_j \frac{\partial W}{\partial x_k}\, d\tau - \int W x_j n_k \, dS'\right]$$
$$= -\epsilon_{ijk}\left[u_j \int \phi n_k \, dS + \omega_j \int \phi n_{3+k}\, dS + 4\pi \int \sigma x_j v_k \, d\tau\right] -$$
$$-\epsilon_{ijk}\int (x_j v_k V_l n_l + u_j \phi n_k + \omega_j \phi n_{3+k} + W x_j n_k)\, dS'.$$

Hence, evaluating the integrals over S' by the already frequently applied procedure and putting

$$\epsilon_{ijk}\left[\frac{d}{dt'}\int \phi x_j n_k \, dS + \omega_j \int \phi n_{3+k}\, dS\right] = \epsilon_{ijk}\frac{d}{dt}\int \phi x_j n_k \, dS,$$

the expression for the moment becomes

$$M_i = -\rho\epsilon_{ijk}\left[\frac{d}{dt}\int \phi x_j n_k \, dS + u_j \int \phi n_k \, dS\right] + M_{iL}(v), \tag{63}$$

where $M_{iL}(v)$ denotes the Lagally moment for steady flow,

$$M_{iL}(v) = -4\pi\rho\epsilon_{ijk}\left[\int \sigma x_j v_k \, d\tau + \sum\left(m x_j v_k' + \mu_j v_k' + x_j \mu_l \frac{\partial v_k'}{\partial x_l} - \tfrac{4}{3}\pi\sigma x_j \mu_k\right)\right].$$

This differs from the expressions derived by Lagally and Cummins for steady flow in the appearance of the last term, because they did not consider the case in which distributed sources and isolated doublets are simultaneously present. Except for this term, (63) may be shown to be equivalent to the result derived by Cummins for unsteady flow. Cummins, however, did not succeed in expressing the moment of momentum integral in (63) in terms of singularities, so that its time derivative occurs explicitly in his final result. As in the case of the Lagally force, the contributions to v from internal singularities need not be considered in computing the Lagally moment since they annul each other in summation.

Now, substituting for the momentum and moment of momentum integrals in (63) from (55) and (57a), and noting from (7) that

$$-B\epsilon_{ijk}\frac{d}{dt}(u_j \bar{x}_k) + \epsilon_{ijk} u_j u_\alpha B_{\alpha k} = B\epsilon_{ijk}\bar{x}_j \frac{du_k}{dt},$$

we obtain

$$M_i = B\epsilon_{ijk}\bar{x}_j\frac{du_k}{dt} - \frac{d}{dt}(\omega_j A_{3+\alpha,\,3+j}) - 4\pi\rho\frac{d}{dt}\left\{\int[\sigma_\beta(\phi_0' + u_j x_j) - \sigma_0 \phi_\beta']\, d\tau + \right.$$
$$\left. + \Sigma\,[m_\beta(\phi_0' + u\ x_j) - m_0 \phi_\beta' - \mu_{\beta j}(v_{0j}' - u_j) + \mu_{0j} v_{\beta j}]\right\} + M_{iL}(v-u), \tag{64}$$

where $\beta = 3+i$, and $M_{iL}(v-u)$ denotes the value of the Lagally moment when the velocity at the singularity relative to that of the body is used.

For elongated bodies with nearly ellipsoidal sections, substituting for $A_{3+i,3+j}$ from (49), we can obtain an approximate but complete expression for the moment in terms of singularities. Thus, applying (57b), we have

$$M_3 = B\epsilon_{3jk}\,\bar{x}_j\frac{du_k}{dt} + \frac{d}{dt}(\omega_j B_{3+j,6}) - 4\pi\rho\frac{d}{dt}\left\{\int(\sigma_6\,\Omega_6 - \sigma_0\,\phi_6')\,d\tau + \right.$$

$$\left. +\Sigma\left[m_6\,\Omega_6 - m_0\,\phi_6' + \mu_{6j}\frac{\partial\Omega_6}{\partial x_j} + \mu_{0j}\,v_{6j}'\right]\right\} + M_{iL}(v-u). \quad (64\,\text{a})$$

A similar expression for M_2 can be written down by symmetry. In the application of (64) and (64 a) it will usually be convenient to make the first term vanish either by choosing the origin at a point of zero acceleration, or at the centroid, or at a point whose acceleration vector passes through the centroid.

6. Comparison of the results of Kirchhoff, Taylor and Lagally

Kirchhoff's equations of motion of a body through a fluid express the force and moment acting on a body in terms of its added masses. Since the generalized Taylor theorem expresses the added masses in terms of the singularities, it appears that, by substituting these expressions for the added masses into Kirchhoff's equations, it might be possible to derive identical formulas to those derived above for the forces and moments in terms of the singularities.

First let us consider one of Kirchhoff's equations (Lamb 1932, p. 168), for the case when the fluid is disturbed only by the motion of the body,

$$F_1 = -\frac{d}{dt'}\frac{\partial T}{\partial u_1} + u_6\frac{\partial T}{\partial u_2} - u_5\frac{\partial T}{\partial u_3}, \qquad 2T = A_{\alpha\beta}\,u_\alpha\,u_\beta, \qquad (65)$$

where T is the kinetic energy of the fluid and the prime denotes that the time derivative is taken relative to a moving coordinate system. Then

$$F_1 = -\frac{d}{dt'}(u_\alpha A_{\alpha 1}) - u_\alpha(u_5\,A_{\alpha 3} - u_6\,A_{\alpha 2}) = -\frac{d}{dt}(u_\alpha A_{\alpha 1}). \qquad (66)$$

On examining the expression for the corresponding term in (60), it is seen that the terms in the first bracket vanish since, in the present case, there are no external singularities and consequently no internal images of them. The terms in the second bracket vanish because the mutual contributions of the internal sources and doublets to the force on the body annul each other. Consequently, (60) and (66) are seen to be in agreement.

More generally, if it is supposed that other boundaries are present, but that only the given body is in motion, the force on the body due to the fluid may be obtained by applying Lagrange's equations (Lamb 1932, p. 188) in the form

$$F_i = -\frac{d}{dt}\frac{\partial T}{\partial u_i} + \frac{\partial T}{\partial \xi_i}, \qquad 2T = A_{\alpha\beta}\,u_\alpha\,u_\beta, \qquad (67)$$

where ξ_i $(i=1,2,3)$ are the coordinates at the origin of the coordinate system attached to the body, relative to an inertial system. Then

$$F_i = -\frac{d}{dt}(u_\alpha A_{\alpha i}) + \frac{1}{2}\frac{\partial A_{\alpha\beta}}{\partial \xi_i} u_\alpha u_\beta. \tag{68}$$

Comparison of this result with (60), in which the first bracket vanishes in the present case since the external boundaries are at rest, shows that the two expressions for the force would be identical in form only if

$$\frac{\partial A_{\alpha\beta}}{\partial \xi_i} u_\alpha u_\beta = -8\pi\rho\left[\int \sigma v_i \, d\tau + \sum\left((m v_i' - \tfrac{4}{3}\pi\sigma\mu_i + \mu_j \frac{\partial v_i'}{\partial x_j}\right)\right],$$

or, putting $\sigma = u_\alpha \sigma_\alpha$, $v_i = u_\beta v_{\beta i}$, etc., only if

$$\frac{\partial A_{\alpha\beta}}{\partial \xi_i} = -8\pi\rho\left[\int \sigma_\alpha v_{\beta i} \, d\tau + \sum\left(m_\alpha v_{\beta i}' - \tfrac{4}{3}\pi\mu_{\alpha i}\sigma_\beta + \mu_{\alpha j}\frac{\partial v_\beta'}{\partial x_j}\right)\right]. \tag{69}$$

But substitution of the generalized Taylor formula for the added masses in terms of singularities into (68) does not seem to yield identically the generalized Lagally formula for the force; in fact, such a complete substitution could not be made since not all the added masses can be so expressed. Thus the method of Kirchhoff–Lagrange appears in general to furnish an alternative (and more limited) method of computing the forces on a body. Nevertheless, (69) must be valid, since the force may be obtained by either method, so that we have expressions for the gradients of the added masses in terms of the singularities.

It will be instructive to illustrate both methods by computing the force on a sphere A of radius a moving with velocity u_1 along the line of centres away from a fixed sphere B of radius b. Let c be the distance between centres at a given instant. Using the method of successive images, we begin with a doublet of strength $\mu_0 = \frac{1}{2}u_1 a^3$ at the centre of A. Its first image in B is a doublet of strength

$$\mu_1 = -\frac{b^3}{c^3}\mu_0 = -\frac{u_1 a^3 b^3}{2c^3}$$

at a distance $\xi_1 = c - b^2/c$ from the centre of A. This gives a second image in A, a doublet of strength

$$\mu_2 = -\frac{b^3}{\xi_1^3}\mu_1 = \frac{u_1 a^6 b^3}{2(c^2-b^2)^3}.$$

To this order of approximation, from (36) the added mass A_{11} is

$$A_{11} = -B_{11} + 4\pi\rho\frac{\mu_0+\mu_2}{u_1}, = \tfrac{2}{3}\pi\rho a^3\left[1 + \frac{3a^3b^3}{(c^2-b^2)^3}\right],$$

and (68) gives for the force

$$F_1 = -\frac{d}{dt}(u_1 A_{11}) + \tfrac{1}{2}u_1^2\frac{dA_{11}}{dc} = -\frac{d}{dt}(u_1 A_{11}) - \frac{6\pi\rho u_1^2 a^6 b^3 c}{(c^2-b^2)^4}.$$

In order to apply (60) we also need the velocity v', due to the first doublet image μ_1, along the line of centres,

$$v' = 2\mu_1(x - b^2/c)^{-3},$$

where x is distance measured from the centre of B. Then

$$\frac{\partial v'}{\partial x} = \frac{3u_1\, a^3 b^3 c}{(xc - b^2)^4}.$$

In order to obtain the same order of approximation as above, we need compute only the force on the original doublet. We obtain from (60)

$$F_1 = -\frac{d}{dt}(u_1 A_{11}) - 4\pi\rho\mu_0\left(\frac{\partial v'}{\partial x}\right)_c = -\frac{d}{dt}(u_1 A_{11}) - \frac{6\pi\rho u_1^2\, a^6 b^3 c}{(c^2 - b^2)^4},$$

which agrees with the result above.

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References

Birkhoff, G. 1953 *Hydrodynamics*. Princeton University Press.

Cummins, W. E. 1953 The forces and moments acting on a body moving in an arbitrary potential stream. *David Taylor Model Basin, Rep.* no. 708.

Lagally, M. 1922 Berechnung der Kräfte und Momente die strömende Flüssigkeiten auf ihre Begrenzung ausüben. *Z. angew. Math. Mech.* **2**, 409.

Lamb, H. 1932 *Hydrodynamics*, 6th Ed. Cambridge University Press.

Landweber, L. 1956 On a generalization of Taylor's virtual mass relation for Rankine bodies. *Quart. Appl. Math.* **14**, 51.

Taylor, G. I. 1928 The energy of a body moving in an infinite fluid, with an application to airships. *Proc. Roy. Soc.* A, **120**, 13.